On a Question of Arveson about Ranks of Hilbert modules

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Abstract

It's well known that the functional Hilbert space $\mathcal{H}^2$ over the unit ball $B_d \subset \mathbb{C}^d$, with kernel function $K(z,\omega) = \frac{1}{1-z_1\omega_1-\cdots-z_d\omega_d}$, admits a natural $A(B_d)$-module structure. We show the rank of a nonzero submodule $\mathcal{M} \subset \mathcal{H}^2$ is infinity if and only if $\mathcal{M}$ is of infinite codimension. Together with Arveson's dilation theory, our result shows that Hilbert modules stand in stark contrast with Hilbert basis theorem for algebraic modules. This result answers a question of Arveson.

KEYWORDS: Hilbert module, submodule, rank.

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0. Introduction : In the study of Multivariable Operator Thoery (MOT), there is a natural approach via Hilbert modules. (Readers are refered to [4] for more information.) Let $T = (T_1, \cdots, T_d)$ be a tuple of commuting operators acting on a common separable Hilbert space $\mathcal{H}$. There is a natural $A(B_d)$-module structure on $\mathcal{H}$:

$$f \cdot \xi = f(T_1, \cdots, T_d)\xi, \qquad f \in A(B_d), \quad \xi \in \mathcal{H}.$$

This approach allows us to introduce algebraic techniques to operator theory. This is not so significant in one variable, since function theory in one variable is often mature enough for analyzing problems. But in higher dimensions, the situation is very different. We have to deal with algebraic and analytic varieties, which are discrete points in one variable. Hence some algebraic

tools, like localization, are useful and indispensible. (See [5], [6], [7], [9],$\cdots$ for some nontrivial applications of algebraic techniques in operator theory.)

The purpose of this note is to show a phenomenon of Hilbert modules which is in stark contrast with Hilbert basis theorem for finitely generated modules over Neotherian rings. It suggests topological modules may behave very differently from algebraic modules.

However in [8], we will show that another aspect of algebraic modules, namely the Hilbert polynomial, suits well for Hilbert modules.

1. Preliminaries: a brief review of $\mathcal{H}^2$: In this section, we review briefly the basic facts about the Hilbert modules we are intersted in. Readers can find more information in [1], [2], [3].

Let $B_d \subset \mathbb{C}^d$ be the open unit ball in $\mathbb{C}^d$. Let $\mathcal{H}^2$ be the functional Hilbert space over $B_d \subset \mathbb{C}^d$, determined by the reproducing kernel $K(z,\omega) = \frac{1}{1-z_1\omega_1-\cdots-z_d\omega_d}$, where $z=(z_1,\cdots,z_d), \omega=(\omega_1,\cdots,\omega_d) \in \mathbb{C}^d$. In particular, different monomials in $\mathcal{H}^2$ are orthogonal: $\langle z^I, z^J\rangle = 0, \quad I \neq J$, where $I, J \in \mathbb{Z}_d^+$ are multi-indices. Let $M_z = (M_{z_1},\cdots,M_{z_d})$ be the tuple of multiplication by coordinate functions on $\mathcal{H}^2$. Then M_z is a tuple of commuting, bounded operators. We have

$$I - M_{z_1}{M_{z_1}}^* - \cdots - M_{z_d}{M_{z_d}}^* = P_0,$$

where P_0 is the projection onto the constant term in $\mathcal{H}^2$. Except for P_0, we will use $P_{(L)}$ denoting the orthogonal projection with the range L.

For any tuple $T=(T_1,\cdots,T_d)$ of commuting bounded operators on $\mathcal{H}$, following Arveson's language, we say T is d-contractive if

$$I - T_1T_1^* - \cdots - T_dT_d^* \geq 0.$$

So it follows $M_z = (M_{z_1},\cdots,M_{z_d})$ is d-contractive. For any d-contractive operator tuple $T=(T_1,\cdots,T_d)$ on $\mathcal{H}$, the *"defect operator"* is defined to be

$$\Delta = (I - T_1T_1^* - \cdots - T_dT_d^*)^{\frac{1}{2}}.$$

The rank of $T=(T_1,\cdots,T_d)$ is defined to the dimension of $\overline{\Delta\mathcal{H}}$. See [1] for more information on this definition. By the dilation theory in [1], for any d-contractive tuple $T=(T_1,\cdots,T_d)$ on $\mathcal{H}$, there exists an auxiliary Hilbert space $\mathcal{L}$, a spherical isometry $S=(S_1,\cdots,S_d)$ on $\mathcal{L}$, (*i.e.* $S_1S_1^* + \cdots +$

$S_dS_d^* = I$), and coinvariant subspaces $\mathcal{K} \subset \mathcal{H}^2 \otimes \mathbb{C}^{(rank(T))}$ and $\mathcal{K}' \subset \mathcal{L}$ such that

$$T \cong Pr_{(\mathcal{K})}(M_z \otimes I_{rank(T)}) \oplus Pr_{(\mathcal{K}')}(S),$$

where the coinvariant subspaces are joint coinvariant with respect to obvious actions. Since it is usually clear from the context which operators we are talking about, we will use the term "(co-)invariant" without specifying the actions. When the second term $Pr_{(\mathcal{K}')}(S)$ is null, T is called a *"pure"* d-contraction.

Let $R_{z_j} = M_{z_j}|_{\mathcal{M}}$ $(1 \leqslant j \leqslant d)$ be the restrictions of M_{z_j} $(1 \leqslant j \leqslant d)$ to the submodule $\mathcal{M}$. Because

$$\begin{aligned} \sum_{j=1}^{d} P_{\mathcal{M}} M_{z_j} P_{\mathcal{M}} \cdot P_{\mathcal{M}} M_{z_j}^* P_{\mathcal{M}} \;&\leqslant\; \sum_{j=1}^{d} P_{\mathcal{M}} M_{z_j} M_{z_j}^* P_{\mathcal{M}} \\ \leqslant P_{\mathcal{M}} P_{\mathcal{M}} \;&=\; P_{\mathcal{M}}, \end{aligned}$$

$R_z = (R_{z_1}, \cdots, R_{z_d})$ is d-contractive. If the spherical isometry part of R is not null, it implies R_z has a nontrivial reducing subspace. This reducing subspace will also be a reducing subspace of $M_z = (M_{z_1}, \cdots, M_{z_d})$ on $\mathcal{H}^2$. It is impossible. So $R_z = (R_{z_1}, \cdots, R_{z_d})$ is "pure."

The rank of submodule $\mathcal{M}$ is defined to be the rank of the tuple of restrictions $R_z = (R_{z_1}, \cdots, R_{z_d})$. The main result of this note is

Theorem 1 (Main Result) *Let $\mathcal{M} \subset \mathcal{H}^2$ be a nonzero submodule. Then $\mathcal{M}$ has rank infinity if $\mathcal{M}$ is of infinite codimension.*

Remark:(1) If submodule $\mathcal{M}$ is of finite codimension, it is easy to see the rank of $\mathcal{M}$ is finite.
(2)When $\mathcal{M}$ is finitely generated by homogeneous polynomials, the above result is also proved in [3].
(3) The above result should be compared with the corresponding result in commutative algebra. Assume R is a Noetherian ring. (*i.e.* R is a commutative ring with unit e and any ideal of R admits a finite system of generators.) Then $F = R$ will be the free module of rank one. Assume $M \subset F^{(n')}$ is a submodule of the free module $F^{(n')}$. Then by Hilbert basis theorem, M admits a finite system of generators $\{g_1, \cdots, g_n\}$. For the free module $F^{(n)}$ with finite rank n, there exists a unique surjective module map $\varphi : F^{(n)} \to M$ which

maps the canonical basis vector e_i to g_i $(1 \leqslant i \leqslant n)$. Let $N = ker(\varphi) \subset F^{(n)}$. Then N is a submodule of the free module $F^{(n)}$ such that

$$0 \to N \to F^{(n)} \to M \to 0$$

is a short exact sequence.

Recall that in [3], $\mathcal{H}^2$ is justified to be the free module of rank one in the category of pure d-contractive Hilbert modules. By the dilation theory in [1], if $\tilde{M}$ is a pure d-contractive Hilbert module with rank $\tilde{n}$, then there exists a submodule $\tilde{N} \subset \mathcal{H}^2 \otimes \mathbb{C}^{(\tilde{n})}$, such that $\tilde{N}^{\perp}$ is unitarily equivalent to $\tilde{M}$ as pure d-contractive modules. Or, we have the following short exact sequence

$$0 \to \tilde{N} \to \mathcal{H}^2 \otimes \mathbb{C}^{(\tilde{n})} \to \tilde{M} \to 0.$$

If $\tilde{M} \subset \mathcal{H}^2$ is a submodule of the free module $\mathcal{H}^2$ with infinite codimension, by dilation theory in [1] and the above theorem, we cannot find a finite integer $\tilde{n}$, such that $\tilde{M}$ is unitarily equivalent to $\tilde{N}^{\perp}$ for some submodule $\tilde{N} \subset \mathcal{H}^2 \otimes \mathbb{C}^{(\tilde{n})}$. Or, we cannot make the above short sequence of Hilbert modules exact with some finite $\tilde{n}$.

Corollary 2 *Let $\mathcal{M} \subset \mathcal{H}^2$ be a nonzero submodule. Then there exists some integer $n \in \mathbb{N}$ and some submodule $\mathcal{N} \subset \mathcal{H}^2 \otimes \mathbb{C}^{(n)}$, such that $\mathcal{M}$ is unitarily equivalent to $\mathcal{N}^{\perp}$ as pure d-contractive Hilbert modules if and only if $\mathcal{M}$ is of finite codimension in $\mathcal{H}^2$.*

2. Jumping operators $\mathbf{J_{z_j}} : \mathcal{M}^{\perp} \to \mathcal{M} \quad (\mathbf{1} \leqslant \mathbf{j} \leqslant \mathbf{d})$: First we decompose M_{z_j} $(1 \leqslant j \leqslant d)$ with respect to $\mathcal{M}$ and $\mathcal{M}^{\perp}$ as :

$$M_{z_j} = \begin{pmatrix} R_{z_j} & J_{z_j} \\ 0 & S_{z_j} \end{pmatrix}, \quad (1 \leqslant j \leqslant d).$$

Most studies of submodules and quotient modules center around R_{z_j} and S_{z_j} $(1 \leqslant j \leqslant d)$ so far. Less attention has been devoted to the "Jumping operators" $J_{z_j} : \mathcal{M}^{\perp} \to \mathcal{M}$ $(1 \leqslant j \leqslant d)$. Our study in this note is closely connected to $J_{z_j} : \mathcal{M}^{\perp} \to \mathcal{M}$ $(1 \leqslant j \leqslant d)$. In fact we find these operators have more significance when we study Fredholm indices, or cohomology groups of a Koszul complex.

Observe that

$$
\begin{aligned}
P_0 &= I - \sum_{j=1}^{d} M_{z_j} M_{z_j}^* \\
&= \begin{pmatrix} I - \sum_{j=1}^{d} R_{z_j} R_{z_j}^* - \sum_{j=1}^{d} J_{z_j} J_{z_j}^* & \cdots \\ \cdots & \cdots \end{pmatrix}.
\end{aligned}
$$

We use $Pr_{(\mathcal{M})}(P_0)$ denoting the compression of P_0 onto subspace $\mathcal{M}$. Since $Pr_{(\mathcal{M})}(P_0)$ is of rank one, we know $rank(I - \sum_{j=1}^{d} R_{z_j} R_{z_j}^*) < \infty$ if and only if $rank(\sum_{j=1}^{d} J_{z_j} J_{z_j}^*) < \infty$. This is equivalent to say $rank(J_{z_j}) < \infty$ for every j $(1 \leqslant j \leqslant d)$. So it suffices to prove the following characterization of infinite codimensional submodule $\mathcal{M} \subset \mathcal{H}^2$.

Theorem 3 *Let $\mathcal{M} \subset \mathcal{H}^2$ be a nonzero submodule. Then $\mathcal{M}$ is of infinite codimension if and only if*

$$rank(J_{z_1}) + \cdots + rank(J_{z_d}) = \infty$$

Remark: We will also prove the above result for the Hardy space $\mathcal{H}^2(\mathbb{D}^d)$ in the last section of this note. The proof relies on Beurling's theorem and the fact that $R_z = (R_{z_1}, \cdots, R_{z_d})$ are all isometries. But it seems the idea for the Hardy space $\mathcal{H}^2(\mathbb{D}^d)$ will carry over to $\mathcal{H}^2$, provided we have a good understanding of the invariant subspaces of $\mathcal{H}^2$. It is interesting to compare the proof of analogous results for $\mathcal{H}^2(\mathbb{D}^d)$ and $\mathcal{H}^2$. Since our ultimate goal is to develop a theory for commutting operator tuples, we would like to see a proof of the main result in this note via analyzing the behavior of invariant subspaces of $\mathcal{H}^2$. Especially, we would like to see some results for $\mathcal{H}^2$, which are analogous to Lax-Halmos theorem for Hardy spaces.

3 : Two lemmas : We introduce two lemmas needed for the next section. Moreover, we will use these two lemmas for the Hardy space $\mathcal{H}^2(\mathbb{D}^d)$ in the last section without further explanation since the proofs here carry over to the Hardy space $\mathcal{H}^2(\mathbb{D}^d)$.

Lemma 4 *$dim(\mathcal{M} \ominus z_j \mathcal{M}) = \infty, \quad (1 \leqslant j \leqslant d)$.*

We need to introduce some notation before we give the proof. For any function $f \in \mathcal{H}^2$, let $f = \sum_{n=0}^{\infty} f^{(n)}$ be the homogeneous expansion of f, such

that $f^{(n)}$ is a homogeneous polynomial of degree n. We define

$$ord(f) = inf\{n : \quad f^{(n)} \neq 0, \quad f = \sum_{n=0}^{\infty} f^{(n)}\}.$$

For a submodule $\mathcal{M} \subset \mathcal{H}^2$, we define

$$ord(\mathcal{M}) = inf\{ord(f) : \quad f \in \mathcal{M}\}.$$

Let $I = (z_1, \cdots, z_d)$ be the maximal idea of $A(\mathcal{D}^d)$ at the origin. Then it is easy to see $ord(\overline{I^k \mathcal{M}}) = ord(\mathcal{M}) + k$ $(k \in \mathbb{N})$ (since $ord(\cdot)$ is upper semi-continous).

Proof: We fix some j $(1 \leqslant j \leqslant d)$ first.

Assume $dim(\mathcal{M} \ominus z_j \mathcal{M}) = k < \infty$. We first show

$$dim(\mathcal{M} \ominus z_j^l \mathcal{M}) \leqslant l \cdot k, \qquad l \in \mathbb{N}.$$

It is easy to see

$$\begin{aligned} dim(\mathcal{M} \ominus z_j^l \mathcal{M}) &= dim(\mathcal{M} \ominus z_j \mathcal{M}) + dim(z_j \mathcal{M} \ominus z_j^2 \mathcal{M}) + \\ & \qquad \cdots + dim(z_j^{l-1} \mathcal{M} \ominus z_j^l \mathcal{M}). \end{aligned}$$

For each $t = 1, 2, \cdots, l$, we define $B_t : z_j^{t-1} \mathcal{M} \ominus z_j^t \mathcal{M} \to \mathcal{M} \ominus z_j \mathcal{M}$ by

$$B_t(z_j^{t-1} x) = P_{(\mathcal{M} \ominus z_j \mathcal{M})}(x), \qquad z_j^{t-1} x \in z_j^{t-1} \mathcal{M} \ominus z_j^t \mathcal{M},$$

where $x \in \mathcal{M}$.

If $B_t(z_j^{t-1} x) = 0$, $x \in \mathcal{M}$ and $P_{(\mathcal{M} \ominus z_j \mathcal{M})}(x) = 0$ will imply $x \in z_j \mathcal{M}$. Hence $z_j^{t-1} x \in z_j^t \mathcal{M}$. It follows that B_t is injective, and $dim(\mathcal{M} \ominus z_j^l \mathcal{M}) \leqslant lk$.

Assume $ord(\mathcal{M}) = a$. Then $ord(f) = a$ for some $f \in \mathcal{M}$. We claim

$$dim(\mathcal{M} \ominus I^l \mathcal{M}) \geqslant C_{l+d-1}^d, \qquad l \in \mathbb{N}.$$

(In the above we may also write $\mathcal{M} \ominus \overline{I^l \mathcal{M}}$ since $I^l \mathcal{M}$ is not generally closed. But we want to make the notation simple and do not do that.) For a fixed $l \in \mathbb{N}$, we define

$$L = span\{z^I \cdot f : \quad |I| = i_1 + \cdots + i_d < l\}.$$

Then $P_{(\mathcal{M} \ominus I^l \mathcal{M})}|_L : L \to \mathcal{M} \ominus I^l \mathcal{M}$ is injective.

Otherwise, assume $P_{(\mathcal{M}\ominus I^l\mathcal{M})}(p\cdot f)=0$ for some polynomial p with $deg(p)<l$. Then $ord(p\cdot f)\leqslant l+a-1$. But $P_{(\mathcal{M}\ominus I^l\mathcal{M})}(p\cdot f)=0$ implies $p\cdot f\in\overline{I^l\mathcal{M}}$. So $ord(p\cdot f)\geqslant l+a$. It is impossible.

So it follows

$$dim(\mathcal{M}\ominus I^l\mathcal{M})\geqslant dim(L)\geqslant C^d_{l+d-1},\quad l\in\mathbb{N}.$$

Because $z_j^l\mathcal{M}\subset I^l\mathcal{M}$, we conclude

$$C^d_{l+d-1}\leqslant dim(\mathcal{M}\ominus I^l\mathcal{M})\leqslant dim(\mathcal{M}\ominus z_j^l\mathcal{M})\leqslant l\cdot k,\qquad l\in\mathbb{N}.$$

But when l is large enough, it forces $k=\infty$. $\square$

Remark: By a remarkable theorem due to David Hilbert and [6], the dimension function $\varphi(l)=dim(\mathcal{M}\ominus I^l\mathcal{M})$ will become a polynomial when l is large enough. (See [6], [10] for more information.) This polynomial, named after Hilbert, is well-known to algebraists, but less familar to the operator theorists. In the Hilbert module setting, it has some nice properties and applications. We will pursue this topic in [8].

Let $\mathcal{H}_{z_{i1},\cdots,z_{ik}}\subset\mathcal{H}^2$ $(1\leqslant z_{i1}<\cdots<z_{ik}\leqslant d)$ be the subspace of $\mathcal{H}^2$, consisting of elements involving only constants and variables $z_{i1},\cdots,z_{ik}$. Then $\mathcal{H}_{z_1},\cdots,\mathcal{H}_{z_d}$ are the classical Hardy spaces with variables $z_1,\cdots,z_d$ respectively.

Lemma 5 *A submodule $\mathcal{M}\subset\mathcal{H}^2$ is of finite codimension in $\mathcal{H}^2$ if and only if $\mathcal{M}\cap\mathcal{H}_{z_j}=u_j(z_j)\mathcal{H}_{z_j}$ for some finite nonzero Blaschke product $u_j(z_j)$ in variable z_j, $(1\leqslant j\leqslant d)$.*

Proof: If $dim(\mathcal{M}^\perp)<\infty$, the codimension of $\mathcal{M}\cap\mathcal{H}_{z_j}$ in $\mathcal{H}_{z_j}$ is finite for each j $(1\leqslant j\leqslant d)$. It is well known that an invariant subspace of the classical Hardy space is of finite codimension if and only if the corresponding inner function is a finite Blaschke product.

If $\mathcal{M}\cap\mathcal{H}_{z_j}=u_j(z_j)\mathcal{H}_{z_j}$ for some finite Blaschke product u_j in variable z_j for each j $(1\leqslant j\leqslant d)$, $\mathcal{M}\cap\mathcal{H}_{z_j}$ then contains a nonzero polynomial $p_j(z_j)$. Assume $deg(p_j(z_j))\leqslant A$ for each j $(1\leqslant j\leqslant d)$. Let $\pi:\mathcal{H}^2\to\mathcal{H}^2/\mathcal{M}=\mathcal{M}^\perp$ be the quotient map. Then

$$\mathcal{M}^\perp=span\{\pi(z^I):\quad I=(i_1,\cdots,i_d),i_r<A,1\leqslant r\leqslant d\}$$

is of finite dimension. $\square$

4. Proof of main result for $\mathbf{d = 2}$: We will prove the main result for $d = 2$, and then use induction argument to finish the general case. As we have mentioned, since our goal is to develop MOT, we would like to see a proof without using induction argument.

Let $\mathcal{K} = \mathcal{M}^{\perp}$ and assume $dim(\mathcal{K}) = \infty$. For each j $(1 \leqslant j \leqslant d)$, we have

$$\begin{aligned} \{\mathcal{K} + z_j\mathcal{K} + z_j\mathcal{M}\}^{\perp} &= \mathcal{M} \cap \mathcal{H}_{z_1,\cdots,\hat{z_j},\cdots,z_d} \\ &= (\mathcal{M} \ominus z_j\mathcal{M}) \cap \mathcal{H}_{z_1,\cdots,\hat{z_j},\cdots,z_d}. \end{aligned}$$

We decompose $\mathcal{M} \ominus z_j\mathcal{M}$ $(1 \leqslant j \leqslant d)$ into orthogonal sums as

$$\mathcal{M} \ominus z_j\mathcal{M} = [(\mathcal{M} \ominus z_j\mathcal{M}) \cap \mathcal{H}_{z_1,\cdots,\hat{z_j},\cdots,z_d}] \oplus \mathcal{E}_j,$$

where $\mathcal{E}_j \subset \overline{\mathcal{K} + z_j\mathcal{K} + z_j\mathcal{M}}$, $1 \leqslant j \leqslant d$. So we have

$$\mathcal{E}_j \subset P_{(\mathcal{M}\ominus z_j\mathcal{M})}(z_j\mathcal{K}), \quad 1 \leqslant j \leqslant d.$$

For any $x \in \mathcal{E}_j$ $(1 \leqslant j \leqslant d)$, we may write $x = P_{(\mathcal{M}\ominus z_j\mathcal{M})}(z_j\xi)$ $(1 \leqslant j \leqslant d)$, where $\xi \in \mathcal{K}$. Then $P_{(\mathcal{M}\ominus z_j\mathcal{M})}(J_{z_j}(\xi)) = x$ $(1 \leqslant j \leqslant d)$. It follows

$$rank(J_{z_j}) \geqslant dim(\mathcal{E}_j), \qquad 1 \leqslant j \leqslant d.$$

From now on, we assume

$$dim(\mathcal{E}_j) < \infty, \qquad 1 \leqslant j \leqslant d.$$

This assumption will be used as we finish our proof by induction in the next section.

For the rest of this section, we assume $d = 2$.

By Beurling's theorem, there is an inner function $\varphi(z_2) \in \mathcal{H}_{z_2}$, such that

$$\begin{aligned} (\mathcal{M} \ominus z_1\mathcal{M}) \cap \mathcal{H}_{z_2} &= \mathcal{M} \cap \mathcal{H}_{z_2} \\ &= \varphi(z_2)\mathcal{H}_{z_2}. \end{aligned}$$

By Lemma 4 $dim(\mathcal{M} \ominus z_1\mathcal{M}) = \infty$ and our assumption $dim(\mathcal{E}_1) < \infty$, we know $\varphi(z_2)$ is nonzero.

For any polynomial $p(z_1)$ in variable z_1, we define

$$S_{\{p\}} = \{g \in \mathcal{M} : g = p(z_1)f(z_2), f(z_2) \in Hol(\mathbb{D})\}.$$

Let $ord_{z_2}(f(z_2))$ be the multiplicity of the zeros at the origin of the holomorphic function $f(z_2)$ in variable z_2. For any $h(z_1, z_2) = g(z_1)f(z_2)$, where $g(z_1), f(z_2)$ are holomorphic, we define

$$ord_{z_2}(h(z_1, z_2)) = ord_{z_2}(f(z_2)).$$

For polynomial $p(z_1)$, we define

$$ord_{z_2}(S_{\{p\}}) = inf\{ord_{z_2}(x) : \quad x \in S_{\{p\}}\},$$

if $S_{\{p\}} \neq \{0\}$. For $S_{\{p\}} = \{0\}$, we define $ord_{z_2}(S_{\{p\}}) = +\infty$. Let

$$a = inf\{ord_{z_2}(S_{\{p\}}) : \quad p \in C[z_1]\},$$

where $C[z_1]$ is the polynomial ring in variable z_1. Since

$$S_{\{1\}} = \mathcal{M} \cap \mathcal{H}_{z_2} = \varphi(z_2)\mathcal{H}_{z_2},$$

we know $a < +\infty$. Assume $p_0(z_1)f_0(z_2)$ achieves the value a.

Now we check that for any nonzero polynomial $p(z_1)$ in z_1,

$$p(z_1)p_0(z_1)f_0(z_2) \notin z_2\mathcal{M}.$$

Otherwise, for some $p(z_1)$,

$$p(z_1)p_0(z_1)f_0(z_2) = z_2 x, \quad x \in \mathcal{M}.$$

So $f_0(z_2) = z_2\tilde{f}_0(z_2)$ for some holomorphic function $\tilde{f}_0(z_2)$ and $x = p(z_1)p_0(z_1)\tilde{f}_0(z_2) \in \mathcal{M}$. Then $ord_{z_2}(x) = a - 1$. Contradiction.

Recall that

$$\mathcal{M} \ominus z_2\mathcal{M} = [(\mathcal{M} \ominus z_2\mathcal{M}) \cap \mathcal{H}_{z_1}] \oplus \mathcal{E}_2,$$

where $dim(\mathcal{E}_2) < \infty$. Let

$$\mathcal{G} = \{p(z_1)p_0(z_1)f_0(z_2) : \quad p(z_1) \in C[z_1]\}.$$

Then $P_{(\mathcal{E}_2)}|_{\mathcal{G}} : \mathcal{G} \to \mathcal{E}_2$ has finite dimensional range. Assume

$$dim(P_{(\mathcal{E}_2)}(\mathcal{G})) = b \leqslant dim(\mathcal{E}_2) < \infty.$$

So there exists an $N \in \mathbb{N}$ such that

$$dim(P_{(\mathcal{E}_2)}(\{p(z_1)p_0(z_1)f_0(z_2): \quad p(z_1) \in C[z_1], deg(p(z_1)) \leqslant N\})) = b.$$

Hence for any $p(z_1) \in C[z_1]$, with $deg(p(z_1)) > N$, there exists some $\tilde{p}(z_1) \in C[z_1]$, with $deg(\tilde{p}(z_1)) \leqslant N$, such that

$$P_{(\mathcal{E}_2)}((p - \tilde{p})p_0 f_0) = 0.$$

This means

$$(p - \tilde{p})p_0 f_0 \in [(\mathcal{M} \ominus z_2\mathcal{M}) \cap \mathcal{H}_{z_1}] \oplus z_2\mathcal{M}.$$

Hence there exists $\xi(z_1) \in (\mathcal{M} \ominus z_2\mathcal{M}) \cap \mathcal{H}_{z_1}$ and $\eta(z_1, z_2) \in \mathcal{M}$, such that

$$(p(z_1) - \tilde{p}(z_1))p_0(z_1)f_0(z_2) = \xi(z_1) + z_2\eta(z_1, z_2).$$

Let $z_2 = 0$, we know

$$\xi(z_1) = (p(z_1) - \tilde{p}(z_1))p_0(z_1)f_0(0)$$

is a polynomial. But we know

$$(p(z_1) - \tilde{p}(z_1))p_0(z_1)f_0(z_2) \notin z_2\mathcal{M}.$$

So $\xi(z_1) \in (\mathcal{M} \ominus z_2\mathcal{M}) \cap \mathcal{H}_{z_1}$ is a nonzero polynomial.

Similarly, we can show $\mathcal{M}$ contains a nonzero polynomial in z_2. Together with Lemma 5, we finish our proof for $d = 2$. $\square$

5. Proof of main result for $\mathbf{d \geqslant 3}$: Recall again

$$\mathcal{M} \ominus z_j\mathcal{M} = [(\mathcal{M} \ominus z_j\mathcal{M}) \cap \mathcal{H}_{z_1,\cdots,\hat{z_j},\cdots,z_d}] \oplus \mathcal{E}_j,$$

where we assume $dim(\mathcal{E}_j) < \infty$ $(1 \leqslant j \leqslant d)$.

Assume our proof is finished for $(d-1)$ dimensional case.

If for some j $(1 \leqslant j \leqslant d)$, $\mathcal{M} \cap \mathcal{H}_{z_1,\cdots,\hat{z_j},\cdots,z_d}$ is of finite codimension in $\mathcal{H}_{z_1,\cdots,\hat{z_j},\cdots,z_d}$, then $\mathcal{M} \cap \mathcal{H}_{z_s}$ is of finite codimension in $\mathcal{H}_{z_s}$ for any s such that $1 \leqslant s \leqslant d$, $s \neq j$.

So if for any j $(1 \leqslant j \leqslant d)$, $\mathcal{M} \cap \mathcal{H}_{z_1,\cdots,\hat{z_j},\cdots,z_d}$ is of finite codimension in $\mathcal{H}_{z_1,\cdots,\hat{z_j},\cdots,z_d}$, Lemma 5 will imply $\mathcal{M}$ is of finite codimension in $\mathcal{H}^2$.

Now we assume for some j $(1 \leqslant j \leqslant d)$, say $j = d$, $\mathcal{M} \cap \mathcal{H}_{z_1,\cdots,z_{d-1}}$ is of infinite codimension in $\mathcal{H}_{z_1,\cdots,z_{d-1}}$. Assumption $dim(\mathcal{E}_j) < \infty$ $(1 \leqslant j \leqslant d)$ implies $\mathcal{M} \cap \mathcal{H}_{z_1,\cdots,z_{d-1}}$ is a nonempty submodule in $\mathcal{H}_{z_1,\cdots,z_{d-1}}$. Now we may apply induction.

Our proof of the main result is completed. □

6. A analogous result for the Hardy space $\mathcal{H}^2(\mathbb{D}^d)$: In this section, we will prove Theorem 3 for Hardy space $\mathcal{H}^2(\mathbb{D}^2)$ over the bidisk $\mathbb{D}^2$. The reason why we choose bidisk $\mathbb{D}^2$ instead of the polydisk $\mathbb{D}^d$ is only for notational convenience. The proof here relies heavily on Beurling's theorem and the fact that $R_z = (R_{z_1}, \cdots, R_{z_d})$ are isometries. Let (z, ω) be the coordinate functions on $\mathbb{D}^2$. $\mathcal{H}_z$ and $\mathcal{H}_\omega$ have the same meaning as in Section 3. We decompose M_{z_j} $(1 \leqslant j \leqslant d)$ with respect to a submodule $\mathcal{M}$ and $\mathcal{M}^\perp$ in the same way as we did in Section 2:

$$M_{z_j} = \begin{pmatrix} R_{z_j} & J_{z_j} \\ 0 & S_{z_j} \end{pmatrix}, (1 \leqslant j \leqslant d).$$

We will prove

Theorem 6 *A nonzero submodule $\mathcal{M} \subset \mathcal{H}^2(\mathbb{D}^d)$ is of infinite codimension if and only if*

$$rank(J_{z_1}) + \cdots + rank(J_{z_d}) = \infty$$

Proof: In fact we will deal with J_z^* and J_ω^*, instead of J_z and J_ω. First assume $\mathcal{M}$ is of infinite codimension.

Observe that $J_z^*(z \cdot x) = 0$ for any $x \in \mathcal{M}$, so we only have to consider the rank of

$$J_z^* : \mathcal{M} \ominus z\mathcal{M} \to \mathcal{M}^\perp.$$

First we look at the kernel of J_z^* on $\mathcal{M} \ominus z\mathcal{M}$. Let

$$L_z = \{x \in \mathcal{M} \ominus z\mathcal{M} : \quad J_z^*(x) = 0\}$$

Then for any $x \in L_z$, *i.e.* $P_{\mathcal{M}^\perp}(\overline{z}x) = 0$, we have $P_{\mathcal{H}^2}(\overline{z}x) \in \mathcal{M}$. But for any $x \in \mathcal{M} \ominus z\mathcal{M}$, we have $\langle \overline{z}x, \mathcal{M} \rangle = 0$. It follows $P_{\mathcal{H}^2}(\overline{z}x) = 0$, and $x \in \mathcal{H}_\omega$. It is easy to see that if $x \in \mathcal{H}_\omega \cap \mathcal{M}$, then $x \in L_z$. So it follows $L_z = \mathcal{H}_\omega \cap \mathcal{M}$.

If $rank(J_z^*) < \infty$, then there exists a finite dimensional subspace $\tilde{L_z} \subset \mathcal{M} \ominus z\mathcal{M}$ such that

$$\begin{aligned} \mathcal{M} \ominus z\mathcal{M} &= L_z + \tilde{L_z} \\ &= \varphi(\omega)\mathcal{H}_\omega + \tilde{L_z}. \end{aligned}$$

Similarly, if $rank(J_z^*) < \infty$, we have

$$\begin{aligned} \mathcal{M} \ominus \omega\mathcal{M} &= L_\omega + \tilde{L_\omega} \\ &= \psi(z)\mathcal{H}_z + \tilde{L_\omega}. \end{aligned}$$

Here $\varphi(\omega)$ and $\psi(z)$ are inner functions. Because of Lemma 5, we may assume $\varphi(\omega)\mathcal{H}_\omega$ is of infinite codimension in $\mathcal{H}_\omega$.

Consider $\mathcal{M}$ is a z-invariant subspace of $\mathcal{H}^2(\mathbb{D}^2) = \mathcal{H}_z \otimes \mathcal{H}_\omega = \mathcal{H}^2(\mathbb{D}, \mathcal{H}_\omega)$. By Lax-Halmos theorem, we can find a subspace $\mathcal{E} \subset \mathcal{H}_\omega$ with infinite codimension in $\mathcal{H}_\omega$, such that the z-invariant subspace $[\mathcal{E}]_z \subset \mathcal{H}^2(\mathbb{D}^2)$, generated by $\mathcal{E}$, contains the z-invariant subspace $[\tilde{L_z}]_z \subset \mathcal{H}^2(\mathbb{D}^2)$, generated by $\tilde{L_z}$, and contains $L_z = \varphi(\omega)\mathcal{H}_\omega$. Choose any inner function $u(\omega) \in \mathcal{H}_\omega \ominus \mathcal{E}$. Then $u(\omega)\mathcal{H}_z$ is orthogonal to $[\mathcal{E}]_z$, hence orthogonal to $\mathcal{M}$ since $\mathcal{M} \subset [\mathcal{E}]_z$. But $u(\omega)\psi(z) \in \psi(z)\mathcal{H}_\omega \subset \mathcal{M}$. Contradiction.

The other direction is trivial. $\square$

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